Five Year Integrated M.Sc. Examination, 2021-2022

Semester - V

Course: PH-3-5-1

Time: Four Hours (Classical Mechanics) Full Marks: 80

Questions are of values as indicated in the margin

Answer question number **one** and any **four** from the rest

1. Answer any **five** questions:

$5 \times 4 = 20$

(a) Prove that if a particle is subject to a central force only, then its angular momentum is conserved.

(b) Define integrable system following Liouville's Theorem.

(c) Define holonomic and nonholonomic constraints with examples.

(d) Show that the Poisson bracket of the components of angular momentum is $\{L_x, L_y\} = L_z$, where L_i $(i = x, y, z)$ is the i-th component of the angular momentum.

(e) State and prove perpendicular axes theorem for a laminar rigid body.

(f) Define stable, unstable and metastable equilibrium with appropriate diagram.

(g) An inelastic ball of mass m has been thrown vertically upward from the ground at $z = 0$. The initial kinetic energy of the ball is E. Draw the phase space trajectory of the ball after successive bouncing on the ground. Due to the inelastic property it looses energy in each collision with the ground.

2. (a) For a function $f(q, p, t)$, prove that

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}\,.$$

(b) The Hamiltonian of a classical one-dimensional harmonic oscillator is $H = \frac{1}{2}(p^2 + x^2)$ in suitable units. Calculate the total time derivative of the dynamical variable $(p - \sqrt{2}x)$.

(c) Show that the degree of freedom of a rigid body in d space dimensions is $d(d+1)/2$.

(d) Starting from the Lagrangian of a double pendulum determine the normal modes of this system when the amplitudes of the oscillations are small. For simplicity, choose masses and lengths of the individual pendulums are same. Draw the configurations of the double pendulum for different normal modes.

4+4+3+4=15

3. (a) Show that the Lagrange's equations of motion remain form invariant under the transformation :

$$L \to L' = L + \frac{dF(q,t)}{dt},$$

where $F(q, t)$ is a function of generalised coordinates (q_i) and time t.

(b) Starting from Lorentz force law, show that the Lagrangian for a charged particle moving with velocity $\vec{v}$ in presence of electromagnetic field is given by

$$L = \frac{1}{2}m|\vec{v}|^2 - e\phi + e(\vec{v}.\vec{A}),$$

where ϕ and $\vec{A}$ are scalar and vector potential respectively.

(c) Show that the gauge transformation in electromagnetic field maps the L to an equivalent Lagrangian L', where $L' = L + \frac{dF(q,t)}{dt}$ and $F(q, t)$ is a function of generalised coordinates (q_i) and time t.

5+6+4=15

4. (a) Show that Lagrange's equations remain form invariant under coordinate transformation.

   (b) Suppose the Lagrangian of a particle moving in one dimension is given by

$$L = \frac{\dot{x}^2}{2x} - V(x)\,.$$

   Calculate the Hamiltonian for this particle.

   (c) Consider a one-parameter family of maps

$$q_i(t) \to Q_i(s,t) \quad s \in \mathbf{R}$$

   such that $Q_i(0,t) = q_i(t)$. This transformation is said to be a continuous symmetry of Lagrangian L if

$$\frac{\partial}{\partial s} L(Q_i(s,t), \dot{Q}_i(s,t), t) = 0\,.$$

5+5+5=15

5. (a) What is Coriolis force?

   (b) Suppose $F(q_i, Q_i)$ is the generating function for the canonical transformation between (q_i, p_i) and (Q_i, P_i). If $p_i = \frac{\partial F}{\partial q_i}$, prove that $P_i = -\frac{\partial F}{\partial Q_i}$.

   (c) Find out the relation between (q_i, p_i) and (Q_i, P_i) while $F(q_i, Q_i) = q_i Q_i$.

   (d) Find the relation between the constant a and b if the following transformation is to be canonical

$$q = aP^{\frac{1}{2}} \sin Q,\; p = bP^{\frac{1}{2}} \cos Q\,.$$

   Find the value of a and b if after the transformation, the Hamiltonian of the harmonic oscillator is to be a multiple of P.

2+4+3+(3+3)=15

6. (a) Suppose the Lagrangian of a heavy symmetrical top is given by

$$L = \frac{1}{2}A(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{1}{2}C(\dot{\phi}\cos\theta + \dot{\psi})^2 - Mgl\cos\theta,$$

where θ, ϕ and ψ are three Euler's angles. Identify the cyclic coordinates and calculate the constants of motion.

(b) Eliminating $\dot{\phi}$ from the energy expression, show that the total energy can be expressed as a function $\dot{\theta}$ and θ.

(c) Use this energy expression in this form to show that the motion of heavy symmetrical top is confined between two boundary values of θ.

5+5+5=15

7. (a) Derive Hamilton's equation from the least action principle.

(b) The Lagrangian for a simple pendulum is given by $L = \frac{1}{2}ml^2\dot{\theta}^2 - mg(1 - \cos\theta)$. Calculate the generalised momentum. Hence calculate the Poisson's bracket between θ and $\dot{\theta}$ i.e. $\{\theta, \dot{\theta}\}$.

(c) Define kinetic energy of a rigid body in terms of moment of inertia tensor (I_{ij}). From the expression of inertia tensor prove that $I_{ij} = I_{ji}$.

(d) Two masses m each are placed at the point (a, a) and $(-a, -a)$ respectively, and one mass, $2m$ is placed at the point $(a, -a)$ and another mass, $3m$ is placed at $(-a, a)$. Calculate the inertia tensor.

3+4+4+4=15